Meeting 10 — Math 22

Section 10.8.1

Activity 10.8.2

- Complete Activity 10.8.2 and discuss with your group.
- Class discussion.

a, b. $f(r,h) = 2\pi rh + 2\pi r^2$ f: surface area
$g(r,h) = \pi r^2 h = 355$ variable radius, height
g: volume

Solve the system

$\langle 4\pi r + 2\pi h, 2\pi r \rangle = \nabla f = \lambda \nabla g = \lambda \langle 2\pi rh, \pi r^2 \rangle$

$\pi r^2 h = 355$

$$\begin{aligned} 4\pi r + 2\pi h &= \lambda 2\pi rh \\ 2\pi r &= \lambda \pi r^2 \\ \pi r^2 h &= 355 \end{aligned}$$

$\rightarrow 2r + h = \lambda rh$

$\rightarrow 2r = \lambda r^2$

Solve for λ. Note that $r \neq 0$ and $h \neq 0$ since the can has positive volume.

Case 1: $\lambda = 0 \Rightarrow 2\pi r = 0 \Rightarrow r = 0 \Rightarrow$ Volume is 0 $\Rightarrow$ Can't happen

Case 2: $\lambda \neq 0 \Rightarrow r \neq 0$.

Eq. 2 $\Rightarrow \lambda = \frac{2}{r}$

Eq. 1 $\Rightarrow 2r + h = 2h \Rightarrow 2r = h$

Eq. 3 $\Rightarrow 2\pi r^3 = 355 \Rightarrow r = \sqrt[3]{\frac{355}{2\pi}}$

$\Rightarrow h = 2r = 2\sqrt[3]{\frac{355}{2\pi}}$

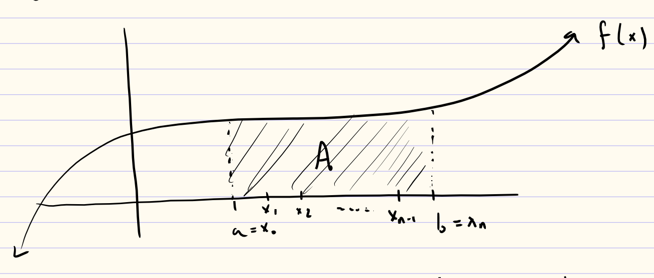
Minimum occurs at $\left(\sqrt[3]{\frac{355}{2\pi}}, 2\sqrt[3]{\frac{355}{2\pi}}\right)$. ∎

End of Chapter 10.

Section 11.1 Double Riemann sums and Double Integrals over Rectangles

Review: Defining the Integral of a Single-variable Function using Riemann Sums.

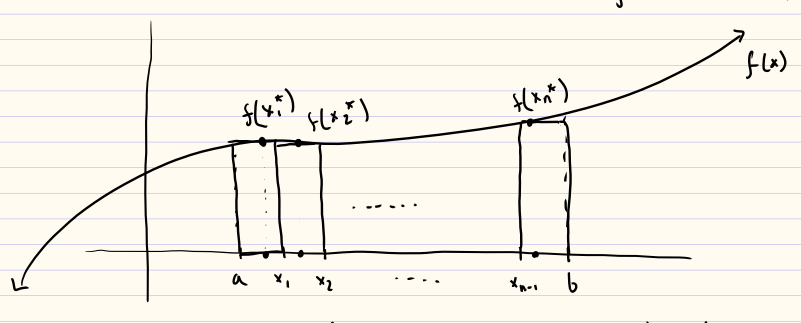
Let $f(x)$ be a function that is continuous on $[a,b]$.



The goal is to compute the area A. To begin, we approximate the area with finitely many rectangles.

Step 1: Partition $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$ with endpoints $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.

Step 2: Within each subinterval $[x_{i-1}, x_i]$, choose a test point x_i^*. Then form rectangles with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$.



Step 3: The area of each rectangle $f(x_i^*)\Delta x$ and so the Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

approximates the area under the curve.

Step 4: Take the number n of rectangles to ∞. This is the Definite Integral

$$\int f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

which, if it does exist, gives the exact area under the curve. ∎

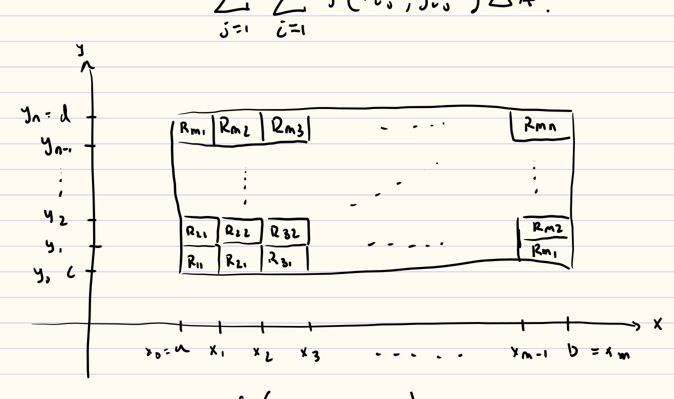
Section 11.1.2 Double Riemann Sums and Double Integrals

Definition 11.1.3 Let f be a continuous function over a rectangle

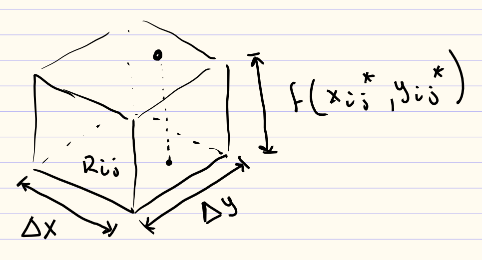
$$R = \{(x,y) : a \le x \le b, c \le y \le d\} = [a,b] \times [c,d].$$

A double Riemann sum of f over R is constructed as follows:

1. Partition $[a,b]$ into m subintervals of length $\Delta x = \frac{b-a}{m}$ with endpoints $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.
2. Partition $[c,d]$ into n subintervals of length $\Delta y = \frac{d-c}{n}$ with endpoints $y_0 = c < y_1 < y_2 < \cdots < y_n = d$.
3. From 1 and 2, we obtain a partition of R into $m \cdot n$ subrectangles
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$
with areas $\Delta A = \Delta x \Delta y$.
4. For each subrectangle R_{ij}, choose a test point (x_{ij}^*, y_{ij}^*) lying in R_{ij}. Then a double Riemann sum is given by
$$\sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$



Each quantity $f(x_{ij}^*, y_{ij}^*)\Delta A$ is the volume of a rectangular prism:



Therefore, the double Riemann sum approximates the signed volume between the graph of f and the xy-plane over R. ∎

Definition 11.1.6 Let R be a rectangle and $f(x,y)$ a function continuous on R. Construct a double Riemann sum for f over R. Then the double integral of f over R is defined to be

$$\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

whenever this limit exists. ∎

Section 11.1.3 Interpretations of Double Riemann Sums & Double Integrals

1. The double Riemann sum approximates signed volume under the graph.
   The double integral measures the exact signed volume.

2. Suppose we want to measure the average value of a function $f(x,y)$ over a rectangle. Partition R as in the definition of double Riemann sum and take $m \cdot n$ sample points (x_{ij}^*, y_{ij}^*). The average value of f among the sample points is given by
$$Avg_{mn} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*).$$

   We have
$$\frac{1}{mn} = \frac{(b-a)(d-c)}{mn} \cdot \frac{1}{(b-a)(d-c)} = \frac{\Delta x \Delta y}{Area(R)} = \frac{\Delta A}{Area(R)}$$

   Therefore
$$Avg_{mn} = \frac{1}{Area(R)} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

   which is a Riemann sum. Taking limits, the exact average value of f over R is given by

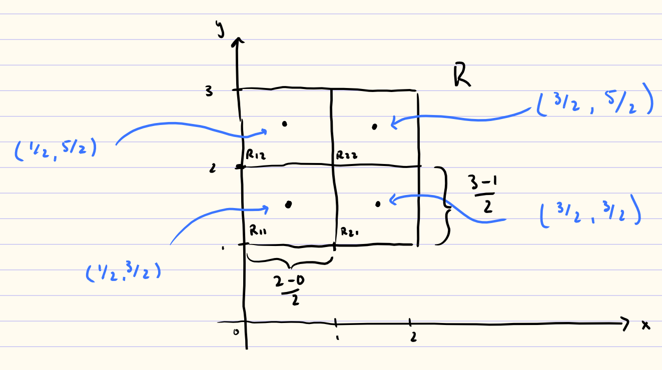
$$f_{AVG} = \lim_{m,n\to\infty} Avg_{mn} = \frac{1}{Area(R)} \iint_R f(x,y)\,dA.$$ ∎

Activity 11.1.3

- Complete Activity 11.1.3 and discuss with your group.
- Class discussion.

Let $f(x,y) = x + 2y$ and $R = [0,2] \times [1,3]$.

a.



c. $\Delta A = \Delta x \Delta y = 1 \cdot 1 = 1$

Value of Riemann Sum: $f(1/2, 3/2) + f(3/2, 3/2) + f(1/2, 5/2) + f(3/2, 5/2) = 20.$

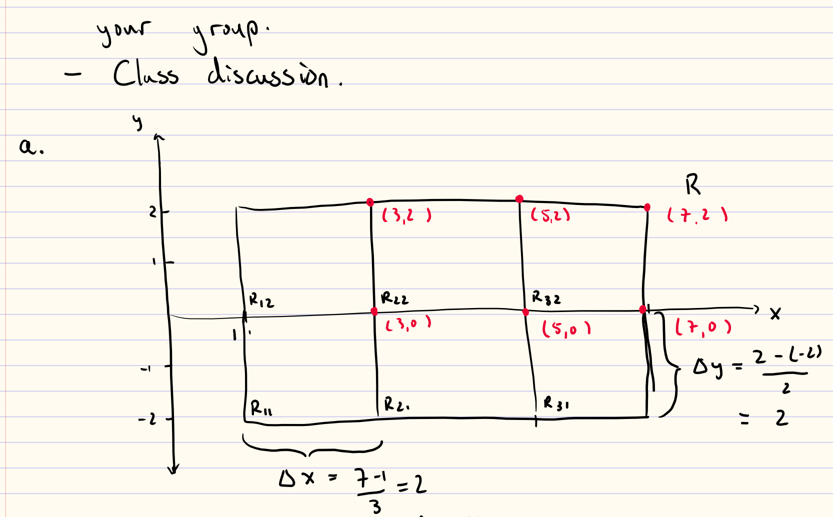
d. The volume between the graph of f and the xy-plane over R is approximately 20.

The average value of f over R is approximately $20/Area(R) = 20/4 = 5$. ∎

Activity 11.1.4

- Complete Activity 11.1.4 and discuss with your group.
- Class discussion.

a.



b. So $\Delta A = 4$. The value of the sum is:

$(2 + 2 + 2 + 0 + 0 + 0) \cdot 4 = 24.$

Table 11.1.8. *Table of values of $f(x,y) = \sqrt{4 - y^2}$.*

	−2	−1	0	1	2
1	0	$\sqrt{3}$	2	$\sqrt{3}$	0
2	0	$\sqrt{3}$	2	$\sqrt{3}$	0
3	0	$\sqrt{3}$	2	$\sqrt{3}$	0
4	0	$\sqrt{3}$	2	$\sqrt{3}$	0
5	0	$\sqrt{3}$	2	$\sqrt{3}$	0
6	0	$\sqrt{3}$	2	$\sqrt{3}$	0
7	0	$\sqrt{3}$	2	$\sqrt{3}$	0

c. Exact value of $\iint_R f(x,y)\,dA$

$$= \frac{\pi \cdot 2^2}{2} \cdot 6 = 12\pi \approx 36$$

To obtain a better approximation, use a mix of test points and use more subdivisions. These are both constrained by the data we are given. ∎